

On vertical boundary layers in a rapidly rotating gas

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The effect of compressibility on vertical boundary layers, of thickness $E^{\frac{1}{2}}$ and $E^{\frac{1}{4}}$ respectively (see Stewartson 1957), in a rapidly rotating gas of constant temperature is studied. Such layers have been considered by Nakayama & Usui (1974). However, these authors used a power-series expansion of the basic density field in their calculations, which means that their results are not uniformly valid in the boundary-layer co-ordinate. In this paper, a uniformly valid asymptotic solution and a numerical solution for the velocity field in each layer are computed. The asymptotic solutions are valid for scale heights of the basic density field large in terms of the reference length for the respective boundary layer. The agreement between the asymptotic and numerical solutions is shown to be good. Although the limits considered have no meaning for density scale heights of the order of the respective boundary-layer thickness, one can formally find an exact solution for the $E^{\frac{1}{4}}$ -layer for any density scale height. In the limit of large density scale height this solution agrees to third order with the corresponding asymptotic solution. It is shown that for both layers the boundary-layer thickness increases (decreases) if the basic density field decreases (increases) in the direction of the respective boundary-layer co-ordinate.

1. Introduction

The flow to be discussed is that of a viscous, thermally conducting, ideal gas contained in a straight axisymmetric container with a flat top and bottom. The container is rotating around its axis of symmetry. The cross-section may be circular or annular. The rate of rotation is assumed to be large, so that effects of gravity and axial stratification are negligible in comparison with that of the radial stratification. For rigid rotation of a gas of constant temperature under such circumstances the density field is given by

$$\rho_0 = \exp\left\{\frac{1}{2}\gamma M^2[(r/r_0)^2 - 1]\right\}, \quad (1)$$

where the reference density is that at the outer periphery, r_0 = outer radius of container divided by half its height, r = radial position divided by half the container height, γ = ratio of specific heats at constant pressure and volume and M = peripheral Mach number. In the following, linear axisymmetric and time-independent perturbations from the isothermal state of rigid rotation are considered, and the structure of the vertical boundary layers is investigated in some

detail. The perturbation flow is assumed to be caused by differential rotation of the top and bottom. Qualitatively similar effects can be obtained by differential heating of the top and bottom (Sakurai & Matsuda 1974). For simplicity, the container walls are assumed to be kept, by external means, at the same temperature as the gas in its unperturbed state. Extension of the results given below to shear layers in the interior of the container is straightforward (cf. Nakayama & Usui 1974). Also, the methods used can be directly carried over to the somewhat more complicated boundary layers occurring when mass is injected and removed at vertical boundaries (Greenspan 1968, p. 106).

2. Fundamental equations

A cylindrical co-ordinate system $\{r, \phi, z\}$ is introduced with the z axis assumed to coincide with the axis of rotation and the lengths r and z non-dimensionalized with half the container height. The top and the bottom are assumed to be at $z = \pm 1$. A detailed derivation of the equations in this section was given, in a somewhat different notation, by Sakurai & Matsuda (1974) and Nakayama & Usui (1974). If starred variables are the dimensional perturbation quantities, suitable definitions for the non-dimensional variables are

$$\begin{aligned} \mathbf{u} &= (u, v, w) = \mathbf{u}^*/\epsilon\Omega H \quad (\text{velocity}), \\ p &= p^*/\epsilon\Omega^2 H^2 \rho_0(r_0) \quad (\text{pressure}), \\ \rho &= \rho^*/\epsilon\rho_0(r_0) \quad (\text{density}), \\ T &= T^*/\epsilon T_0 \quad (\text{temperature}), \end{aligned}$$

where ϵ is the Rossby number, Ω the angular velocity of the basic rigid rotation, $2H$ the height of the container and T_0 the temperature of the basic isothermal state. In the following T_0 is set equal to unity. The Ekman number, which is assumed to be small, is defined as

$$E = \mu/\rho_0(r_0) \Omega H^2,$$

where μ is the dynamic shear viscosity. Viscous effects due to pure dilatation are neglected.

The non-dimensional equations for the perturbation flow can then be written as

$$-2\rho_0 v - r\rho + \partial p/\partial r = E(\nabla^2 - r^{-2})u, \quad (2a)$$

$$2\rho_0 u = E(\nabla^2 - r^{-2})v, \quad (2b)$$

$$\partial p/\partial z = E\nabla^2 w, \quad (2c)$$

$$\frac{1}{r} \frac{\partial}{\partial r} (r\rho_0 u) + \rho_0 \frac{\partial w}{\partial z} = 0, \quad (2d)$$

$$-4\alpha^2 r\rho_0 u = E\nabla^2 T, \quad (2e)$$

$$p = (r_0^2/\gamma M^2) [\rho + \rho_0 T], \quad (2f)$$

where

$$\alpha^2 = \sigma(\gamma - 1) M^2/4r_0^2, \quad \sigma = \mu\gamma c_v/k.$$

The Prandtl number σ is assumed to be of order unity, c_v is the specific heat at constant volume and k is the thermal diffusivity. The first three of these equations are the momentum equations in the radial, tangential and axial directions. Equation (2*d*) is the continuity equation and (2*e*) is the heat conduction equation. The last equation is the equation of state.

3. Brief discussion of the perturbed flow

If the density scale height is not too small, in which case viscous and diffusive flow occurs in the interior of the container, it is reasonable to assume that the flow away from the solid boundaries has a geostrophic character (Howard 1973, private communication; Sakurai & Matsuda 1974; Nakayama & Usui 1974). This interior flow is given by the limiting solutions of (2*a-f*) when the Ekman number approaches zero and was computed by Sakurai & Matsuda (1974) for small Mach number and differential heating of the container walls. These authors showed that the interior swirl velocity is not independent of the vertical co-ordinate as in the case of a homogeneous fluid. Sakurai & Matsuda (1974) and Nakayama & Usui (1974) also computed the Ekman layers at the horizontal boundaries needed to satisfy the no-slip and thermal boundary conditions. At vertical boundaries, one must also construct boundary-layer solutions in order to satisfy the boundary conditions. These layers must also take care of the radial mass transport, of order $E^{\frac{1}{2}}$, induced by Ekman-layer pumping at the top and bottom. In the incompressible case this is accomplished, by two overlapping boundary layers of thicknesses $E^{\frac{1}{2}}$ and $E^{\frac{1}{2}}$ (Stewartson 1957). The $E^{\frac{1}{2}}$ -layer in the compressible case was considered by Sakurai & Matsuda (1974) for the case when the ratio of the density scale height to the boundary-layer thickness is infinitely large. Nakayama & Usui (1974) considered both the $E^{\frac{1}{2}}$ - and the $E^{\frac{1}{2}}$ -layer, but their results are not valid far away from the wall. It is the purpose of the present work to investigate what happens when the scale height of the basic density field is large, but finite, compared with $E^{\frac{1}{2}}$ and/or $E^{\frac{1}{2}}$. For such a situation it seems to be reasonable to assume that the mechanics of the side-wall layers, although very complicated, are only moderately affected by compressibility. We thus assume that the same limits of the governing equations as in the incompressible case have physical meaning and look for solutions of the boundary-layer equations when the ratio of density scale height to boundary-layer thickness is large but finite.

4. Outline of the calculations

In §5 the swirl velocity in the $E^{\frac{1}{2}}$ -layer is considered. Both an exact and an asymptotic multiple-scale solution are obtained. It is shown that these solutions agree to third order when the density scale height becomes large. In §7 the computation of a numerical solution is described. The reason for computing a numerical solution is that for the $E^{\frac{1}{2}}$ -layer no exact solution can be found for comparison with the asymptotic one. However, the results for the $E^{\frac{1}{2}}$ -layer show that the numerical solutions described in §7 are very likely to be excellent

approximations to the corresponding exact solutions for the type of boundary-layer problem considered. In §6 an asymptotic solution for the stream function for the meridional flow in the $E^{\frac{1}{2}}$ -layer is computed. The agreement between this solution and the corresponding numerical one is shown to be good even for rather moderate scale heights of the basic density field. Comparisons are also made with the solutions given by Nakayama & Usui (1974).

5. The $E^{\frac{1}{2}}$ -layer

In the usual way, a stretched variable η is defined as

$$\eta = (r_0 - r)/E^{\frac{1}{2}}.$$

At the inner boundary, if any, a trivial redefinition of η has to be made. As a very thin region is considered, it is legitimate, to first order, to neglect the quadratic variation in the exponent in the formula (1) for the basic density field, and the following approximation can be used:

$$\rho_0 = \exp(\delta\eta), \quad (3)$$

where

$$\delta = \pm 4\alpha^2\gamma r_0 E^{\frac{1}{2}}/\sigma(\gamma - 1) \quad (4)$$

and where the plus sign is to be used at an inner boundary and the minus sign at an outer. A meaningful perturbation problem can be formulated if the dependent variables are assumed to be of the following orders of magnitude:

$$u = O(E^{\frac{1}{2}}), \quad v = O(1), \quad w = O(E^{\frac{1}{2}}), \quad p = O(E^{\frac{1}{2}}), \quad T = O(1), \quad \rho = O(1). \quad (5a-f)$$

For each of these quantities an expansion in powers of $E^{\frac{1}{2}}$ with the leading term scaled according to (5a-f) is assumed. The system of equations for the leading terms in these expansions follows directly from (2a-f) and reads

$$-2\rho_0 v - r_0 \rho - \partial p / \partial \eta = 0, \quad (6a)$$

$$2\rho_0 u = \partial^2 v / \partial \eta^2, \quad \partial p / \partial z = 0, \quad (6b, c)$$

$$-\frac{\partial}{\partial \eta}(\rho_0 u) + \rho_0 \frac{\partial w}{\partial z} = 0, \quad (6d)$$

$$-4\alpha^2 r_0 \rho_0 u = \partial^2 T / \partial \eta^2, \quad (6e)$$

$$\rho + \rho_0 T = (\delta/r_0)p. \quad (6f)$$

From (6b) and (6c) one finds, after eliminating u and integrating twice with respect to η ,

$$T + 2\alpha^2 r_0 v = 0, \quad (6g)$$

where the constants of integration have been equated to zero in view of the boundary-layer character of v and T . Accordingly, the following boundary condition is to be imposed on v and T at $\eta = 0$:

$$T + \langle T_I \rangle + 2\alpha^2 r_0 (v + \langle v_I \rangle) = 0, \quad (6h)$$

where T_I and v_I are the interior temperature and swirl velocity evaluated at $r = r_0$ and the angular brackets stand for vertical averages.

The boundary conditions at the top and the bottom are given by the Ekman-suction formulae, which for this flow have been derived by Howard (1973, private communication).† For the $E^{\frac{1}{2}}$ -layer, the Ekman-suction formulae give, to first order,

$$w = \pm \frac{1}{2}\rho_0^{-1}\kappa^2\partial(\rho_0^{\frac{1}{2}}v)/\partial\eta, \tag{7}$$

where $\kappa^2 = (1 + \alpha^2 r_0^2)^{\frac{1}{2}}$ and the plus sign is to be chosen at the top and the minus sign at the bottom. Combination of (6*b*) and (6*d*) gives

$$\partial^3 v/\partial\eta^3 = 2\rho_0\partial w/\partial z. \tag{8a}$$

From (6*a-f*) it follows that $\partial w/\partial z$ is a function of η only. This function can be found by application of the boundary conditions (7), which yields

$$\partial w/\partial z = \frac{1}{2}\kappa^2\rho_0^{-1}\partial(\rho_0^{\frac{1}{2}}v)/\partial\eta. \tag{8b}$$

Equations (8*a, b*) give, after one integration,

$$\partial^2 v/\partial\eta^2 - \kappa^2 e^{\frac{1}{2}\delta\eta}v = 0. \tag{9}$$

The constant of integration is determined from the requirement that v shall have exponential decay for arbitrarily small δ .

In order to investigate the general structure of the $E^{\frac{1}{2}}$ -layer there is, in view of (6*g, h*), no loss of generality in using the following simplified boundary conditions for v :

$$v(0) = 1, \quad v \rightarrow 0 \quad \text{as} \quad \eta \rightarrow +\infty. \tag{10}$$

The exact solution‡ of (9) is

$$v = c_1 I_0(4\kappa|\delta^{-1}|e^{\frac{1}{2}\delta\eta}) + c_2 K_0(4\kappa|\delta^{-1}|e^{\frac{1}{2}\delta\eta}), \tag{11}$$

where I_0 and K_0 are modified Bessel functions of the first and second kind respectively and c_1 and c_2 are constants. For η large but finite and $|\delta|$ arbitrarily small one finds that

$$v = \begin{cases} K_0(4\kappa\delta^{-1}e^{\frac{1}{2}\delta\eta})/K_0(4\kappa\delta^{-1}), & \delta > 0, \\ I_0(4\kappa|\delta^{-1}|e^{\frac{1}{2}\delta\eta})/I_0(4\kappa|\delta^{-1}|), & \delta < 0, \end{cases} \tag{12a, 12b}$$

are acceptable boundary-layer solutions for vanishing Ekman number, i.e. vanishing δ . Formally, the solution given by (12*a*) has boundary-layer behaviour for δ finite and greater than zero, but this is not the limit under consideration here. For δ fixed and negative, one finds that v is transcendently small in δ for infinitely large η . The asymptotic expansion of (12*a, b*) for small $|\delta|$ and fixed η reads (Erdélyi *et al.* 1953, p. 86)

$$v \sim \frac{\exp\left\{\frac{4\kappa}{\delta}(e^{\frac{1}{2}\delta\eta} - 1)\right\} \left(e^{-\frac{1}{2}\delta\eta} - \frac{\delta}{32\kappa}e^{-\frac{3}{2}\delta\eta} + \frac{9\delta^2}{2048\kappa^2}e^{-\frac{5}{2}\delta\eta} + O(\delta^3 e^{-\frac{7}{2}\delta\eta})\right)}{(1 - \delta/32\kappa + 9\delta^2/2048\kappa^2 + O(\delta^3))}. \tag{13}$$

Of course, this solution reduces to the incompressible case for vanishing δ . Next,

† These formulae can, of course, also be easily derived from the results given by Sakurai & Matsuda (1974) or Nakayama & Usui (1974).

‡ The solution (11) was pointed out to us by Dr S. Johansson.

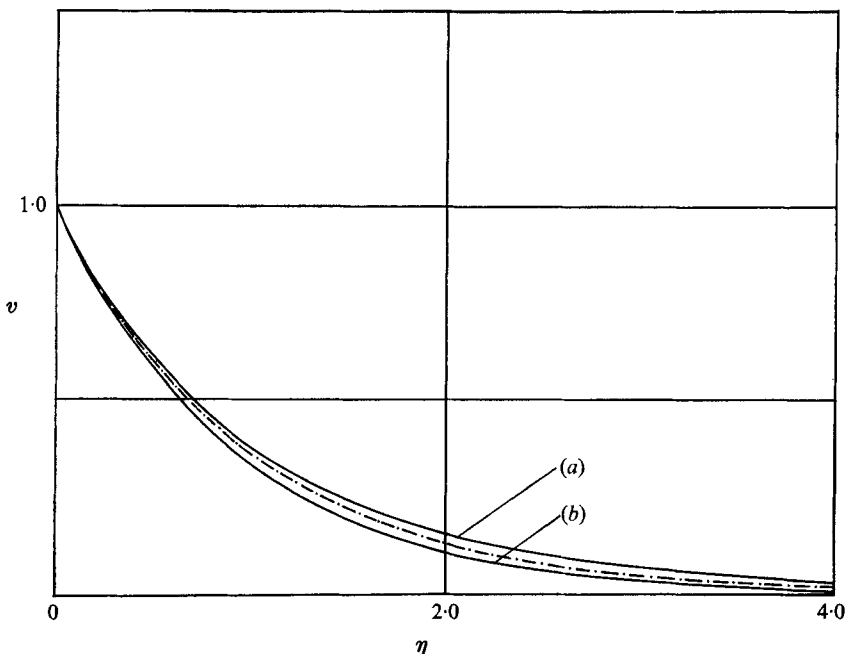


FIGURE 1. $E^{1/2}$ -layer. (a) Two-term Nakayama & Usui (1974) solution, two- and three-term asymptotic solution, numerical solution and exact solution for $\delta = -0.25$. (b) The same as (a) for $\delta = +0.25$. —, incompressible case.

an asymptotic solution is computed directly from (9) subject to (10). Following Cole (1968, p. 102), a 'fast' variable η^+ and a 'slow' variable $\tilde{\eta}$ are defined as†

$$\eta^+ = \int_0^\eta \rho_0^{1/2} d\eta = \frac{4}{\delta} (e^{\delta\eta} - 1), \quad \tilde{\eta} = \delta\eta.$$

η^+ can be viewed as the original variable η distorted by the slow variation of the kinematic viscosity. For the leading terms in the formerly assumed expansion in powers of $E^{1/2}$ another expansion of the form

$$\mathbf{x}(\eta, z; \delta) = \sum_{k=0}^N \delta^k \mathbf{x}_k(\eta^+, \tilde{\eta}, z) + O(\delta^{N+1}), \quad \mathbf{x} = (u, v, w, p, T, \rho), \quad (14)$$

is assumed. Substitution of (14) in (9) gives

$$\frac{\partial^2 v_0}{\partial \eta^{+2}} - \kappa^2 v_0 = 0, \quad (15a)$$

$$\frac{\partial^2 v_1}{\partial \eta^{+2}} - \kappa^2 v_1 = -e^{-\frac{1}{2}\tilde{\eta}} \left(2 \frac{\partial^2 v_0}{\partial \eta^+ \partial \tilde{\eta}} + \frac{1}{4} \frac{\partial v_0}{\partial \eta^+} \right), \quad (15b)$$

$$\frac{\partial^2 v_k}{\partial \eta^{+2}} - \kappa^2 v_k = -e^{-\frac{1}{2}\tilde{\eta}} \left(2 \frac{\partial^2 v_{k-1}}{\partial \eta^+ \partial \tilde{\eta}} + \frac{1}{4} \frac{\partial v_{k-1}}{\partial \eta^+} \right) - e^{-\frac{1}{2}\tilde{\eta}} \frac{\partial^2 v_{k-2}}{\partial \tilde{\eta}^2}, \quad k \geq 2. \quad (15c)$$

† In this case the choice of fast and slow variables is obvious from (13). However, for the $E^{1/2}$ -layer no guidance equivalent to (13) is available.

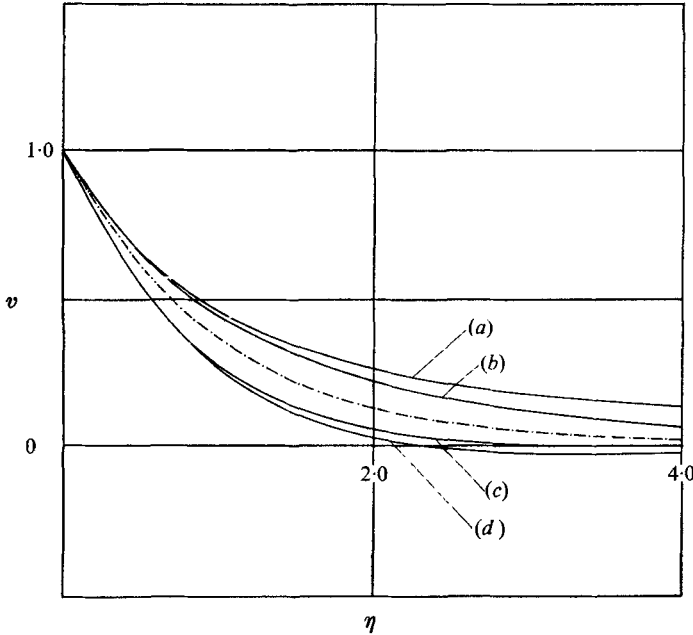


FIGURE 2. E^1 -layer. (a) Two- and three-term asymptotic solution, numerical solution and exact solution for $\delta = -1$. (b) Two-term Nakayama & Usui (1974) solution for $\delta = -1$. (c) The same as (a) for $\delta = +1$. (d) The same as (b) for $\delta = +1$. \cdots , incompressible case.

The physically significant solution of (15a) is

$$v_0(\eta^+, \tilde{\eta}) = \hat{V}_0(\tilde{\eta}) \exp(-\kappa\eta^+),$$

where $\hat{V}_0(\tilde{\eta})$ is determined from the requirement that v_1 has to be non-secular on the η^+ -scale. This implies that the ‘forcing’ term in the equation (15b) for v_1 has to be zero.† Proceeding in this manner to third order, one obtains

$$v = \frac{\left(e^{-\frac{1}{2}\tilde{\eta}} - \frac{\delta}{32\kappa} e^{-\frac{1}{2}\tilde{\eta}} + \frac{9\delta^2}{2048\kappa^2} e^{-\frac{1}{2}\tilde{\eta}} \right) \exp(-\kappa\eta^+)}{1 - \delta/32\kappa + 9\delta^2/2048\kappa^2} + O(\delta^3). \tag{16}$$

This expression agrees with (13).

In figures 1 and 2, the forms $v(\eta)$ according to (12) and (16) are shown for $\delta = \pm 0.25$ and $\delta = \pm 1$. $\alpha^2\gamma_0^2 = 0.2$ in all cases. For comparison solutions obtained by numerical integration (see §7) are also shown. For all these values of δ the exact, numerical and asymptotic (two and three terms respectively) solutions are indistinguishable. The reason for this good agreement for moderate values of δ is that $4\kappa|\delta|^{-1}$ rather than $|\delta|^{-1}$ has to be large for a meaningful asymptotic expansion; see (12). For $\delta = \pm 0.25$ it can be seen that the two-term Nakayama–Usui (1974) solution works excellently and for $\delta = \pm 1$ the error is still not too large, in spite of the fact that this solution was constructed for very small δ .

† Cf. Cole (1968).

6. The $E^{\frac{1}{2}}$ -layer

The stretched variable ξ at the outer boundary is defined as

$$\xi = (r_0 - r)/E^{\frac{1}{2}}. \quad (17)$$

At the inner boundary a trivial redefinition has to be made. The basic density field is given by (3), in which δ now is defined as

$$\delta = \pm 4\alpha^2\gamma r_0 E^{\frac{1}{2}}/\sigma(\gamma - 1),$$

where the plus and minus signs apply at the inner and outer boundary respectively. Taking into account that this layer is transporting a mass flux of order $E^{\frac{1}{2}}$, one finds that a meaningful perturbation problem results if the dependent variables are of the following orders of magnitude:

$$u = O(E^{\frac{1}{2}}), \quad v = O(E^{\frac{1}{2}}), \quad w = O(E^{\frac{1}{2}}), \quad p = O(E^{\frac{1}{2}}), \quad T = O(E^{\frac{1}{2}}), \quad \rho = O(E^{\frac{1}{2}}). \quad (18a-f)$$

Scaling the perturbation quantities according to (18a-f) one finds from (2a-f) the equations

$$-2\rho_0 v - r_0 \rho - \partial p/\partial \xi = 0, \quad (19a)$$

$$2\rho_0 u = \partial^2 v/\partial \xi^2, \quad \partial p/\partial z = \partial^2 w/\partial \xi^2, \quad (19b, c)$$

$$-\partial(\rho_0 u)/\partial \xi + \rho_0 \partial w/\partial z = 0, \quad (19d)$$

$$-4\alpha^2 r_0 \rho_0 u = \partial^2 T/\partial \xi^2, \quad (19e)$$

$$\rho + \rho_0 T = (\delta/r_0) p. \quad (19f)$$

In view of (19d) it is convenient to introduce a stream function ψ defined by

$$\rho_0 u = \partial \psi/\partial z, \quad \rho_0 w = \partial \psi/\partial \xi. \quad (20a, b)$$

Substituting (20a, b) into (19a-e), one obtains after some straightforward calculations

$$2(1 + \alpha^2 r_0^2) \frac{\partial v}{\partial z} + \frac{\partial}{\partial \xi} \left[\rho_0^{-1} \frac{\partial^2}{\partial \xi^2} \left(\rho_0^{-1} \frac{\partial \psi}{\partial \xi} \right) \right] = 0, \quad (21a)$$

$$2\partial \psi/\partial z = \partial^2 v/\partial \xi^2. \quad (21b)$$

From (21a, b) one thus finds a single equation for ψ :

$$\frac{\partial^3}{\partial \xi^3} \rho_0^{-1} \frac{\partial^2}{\partial \xi^2} \rho_0^{-2} \frac{\partial \psi}{\partial \xi} + 4\kappa^4 \frac{\partial^2 \psi}{\partial z^2} = 0. \quad (22)$$

The injection of the radial Ekman-layer flux into the $E^{\frac{1}{2}}$ -layer at the top corner takes place in a square region of area of order E , which in the limit is a point on the $E^{\frac{1}{2}}$ -scale. Therefore, this injection can be modelled by a point source within the present approximation. Similarly, the removal of mass at the bottom corner can be modelled by a point sink. As the axial mass flux, of order $E^{\frac{1}{2}}$, in this layer is too strong to be accepted by a divergent Ekman layer, the stream function must be set to zero on the top and bottom. This gives a non-zero horizontal velocity component at the top and bottom (e.g. see Nakayama & Usui 1974), which necessitates Ekman-layer corrections having a horizontal scale $E^{\frac{1}{2}}$. These correc-

tions are not considered in this work. Furthermore, for simplicity only the part of the flow in the $E^{\frac{1}{2}}$ -layer caused by the Ekman-layer transport is considered. Corrections to the interior flow can be treated in the same manner. Thus the following boundary conditions are imposed on ψ :

$$\psi = 0 \quad \text{for } z = \pm 1, \tag{23a}$$

$$\psi = 1, \quad \partial\psi/\partial\xi = 0 \tag{23b, c}$$

$$\left. \begin{aligned} & \frac{\partial}{\partial\xi} \left(\rho_0^{-1} \frac{\partial^2}{\partial\xi^2} \left(\rho_0^{-1} \frac{\partial\psi}{\partial\xi} \right) \right) = 0 \end{aligned} \right\} \quad \text{for } \xi = 0, \tag{23d}$$

$$\psi \rightarrow 0 \quad \text{as } \xi \rightarrow +\infty. \tag{23e}$$

Equation (23d) follows from (21a) and requiring zero swirl velocity at the boundary. Equation (6g) is valid in this layer too, which means that the boundary condition for the temperature is also fulfilled.

The z dependence is conveniently separated out by putting

$$\psi = \sum_{k=-\infty}^{+\infty} a_k e^{\frac{1}{2}ik\pi z} \phi_k(\xi). \tag{24}$$

If one chooses $\phi_k(0) = 1$ conditions (23a, b) determine the coefficients a_k . One then obtains

$$\frac{d^3}{d\xi^3} \rho_0^{-1} \frac{d^2}{d\xi^2} \rho_0^{-1} \frac{d\phi_k}{d\xi} - \lambda^2 \phi_k = 0, \tag{25}$$

which is to be solved subject to the following boundary conditions:

$$\phi_k = 1, \quad d\phi_k/d\xi = 0 \quad \text{for } \xi = 0, \tag{26a, b}$$

$$\frac{d}{d\xi} \left(\rho_0^{-1} \frac{d^2}{d\xi^2} \left(\rho_0^{-1} \frac{d\phi_k}{d\xi} \right) \right) = 0, \tag{26c}$$

$$\phi_k \rightarrow 0 \quad \text{as } \xi \rightarrow +\infty, \tag{26d}$$

$$\lambda^2 = k^2 \pi^2 \kappa^4.$$

The problem is to find the limiting solutions of (21) when the non-dimensional density scale height δ^{-1} becomes large. The procedure to be used is very similar to the one used for the $E^{\frac{1}{2}}$ -layer. An expansion

$$\phi_k(\xi; \delta) = \sum_{l=0}^N \delta^l \Phi_{kl}(\xi^+, \tilde{\xi}) + O(\delta^{N+1}) \tag{27}$$

is assumed where the ‘fast’ and ‘slow’ variables are defined as

$$\xi^+ = \int_0^\xi \rho_0^{\frac{1}{2}} d\xi = \frac{3}{\delta} (e^{\frac{1}{2}\delta\xi} - 1), \quad \tilde{\xi} = \delta\xi.$$

After some rather lengthy algebra, one finds

$$\left(\frac{\partial^6}{\partial\xi^{+6}} - \lambda^2 \right) \Phi_{k0} = 0, \tag{28a}$$

$$\left(\frac{\partial^6}{\partial\xi^{+6}} - \lambda^2 \right) \Phi_{km} = - \sum_{p=0}^m L_{m-p} \Phi_{kp}, \quad m = 1, \dots, 4, \tag{28b}$$

where

$$L_1 = e^{-\frac{1}{2}\xi} \frac{\partial^5}{\partial \xi^{+5}} \left(6 \frac{\partial}{\partial \xi} - 3 \right),$$

$$L_2 = e^{-\frac{3}{2}\xi} \frac{\partial^4}{\partial \xi^{+4}} \left(15 \frac{\partial^2}{\partial \xi^2} - 20 \frac{\partial}{\partial \xi} + \frac{50}{9} \right),$$

$$L_3 = e^{-\xi} \frac{\partial^3}{\partial \xi^{+3}} \left(20 \frac{\partial^3}{\partial \xi^3} - 50 \frac{\partial^2}{\partial \xi^2} + \frac{110}{3} \frac{\partial}{\partial \xi} - \frac{62}{9} \right),$$

$$L_4 = e^{-\frac{3}{2}\xi} \frac{\partial^2}{\partial \xi^{+2}} \left(15 \frac{\partial^4}{\partial \xi^4} - 60 \frac{\partial^3}{\partial \xi^3} + \frac{245}{3} \frac{\partial^2}{\partial \xi^2} - \frac{376}{9} \frac{\partial}{\partial \xi} + \frac{436}{81} \right).$$

These equations are solved in the same manner as those for the $E^{\frac{1}{2}}$ -layer, i.e. the undetermined functions of ξ^+ appearing in the solution are determined by the requirement that each Φ_{km} has to be non-secular on the ξ^+ scale. After some further algebra, one finds

$$\Phi_{km} = \sum_{k=1}^3 \sum_{l=0}^m C_{kK}^l \frac{f^{l-m}(\xi) \exp(\Omega_{kK} \xi^+)}{(\Omega_{kK})^{m-l}}, \quad m = 0, \dots, 4,$$

where

$$f^0(\xi) = e^{\frac{1}{2}\xi}, \quad f^1(\xi) = -\frac{25}{72} e^{\frac{1}{2}\xi},$$

$$f^2(\xi) = \frac{1369}{10368} e^{-\frac{1}{2}\xi}, \quad f^3(\xi) = -\frac{110807}{2239488} e^{-\frac{1}{2}\xi},$$

$$f^4(\xi) = -\frac{149840305}{644972544} e^{-\frac{1}{2}\xi},$$

$$\Omega_{k1} = -|k\kappa|^{\frac{1}{2}}, \quad \Omega_{k2,3} = -\frac{1}{2}|k\kappa|^{\frac{1}{2}}(1 \pm 3^{\frac{1}{2}} \times i).$$

The constants C_{kK}^0, C_{kK}^1 etc. are successively determined from the following system of equations, corresponding to the boundary conditions (23b-d) at $\xi^+ = \xi^- = 0$:

$$\begin{aligned} \phi_{k0} &= 1, \quad \partial \phi_{k0} / \partial \xi^+ = 0, \quad \partial^4 \phi_{k0} / \partial \xi^{+4} = 0, \\ \phi_{km} &= 0 \quad \text{for } m > 0, \\ \partial \phi_{km} / \partial \xi^+ + \partial \phi_{k(m-1)} / \partial \xi^- &= 0, \\ \frac{\partial^4 \phi_{km}}{\partial \xi^{+4}} + \sum_{p=0}^{m-1} M_{m-p} \phi_{kp} &= 0, \end{aligned}$$

where the operators M_m are defined by

$$M_0 = \frac{\partial^3}{\partial \xi^{+3}} \left(4 \frac{\partial}{\partial \xi} - 2 \right),$$

$$M_1 = \frac{\partial^2}{\partial \xi^{+2}} \left(6 \frac{\partial^2}{\partial \xi^2} - 8 \frac{\partial}{\partial \xi} + \frac{16}{9} \right),$$

$$M_2 = \frac{\partial}{\partial \xi^+} \left(4 \frac{\partial^3}{\partial \xi^3} - 10 \frac{\partial^2}{\partial \xi^2} + \frac{58}{9} \frac{\partial}{\partial \xi} - \frac{20}{27} \right),$$

$$M_3 = \frac{\partial^4}{\partial \xi^4} - 4 \frac{\partial^3}{\partial \xi^3} + 5 \frac{\partial^2}{\partial \xi^2} - 2 \frac{\partial}{\partial \xi}.$$

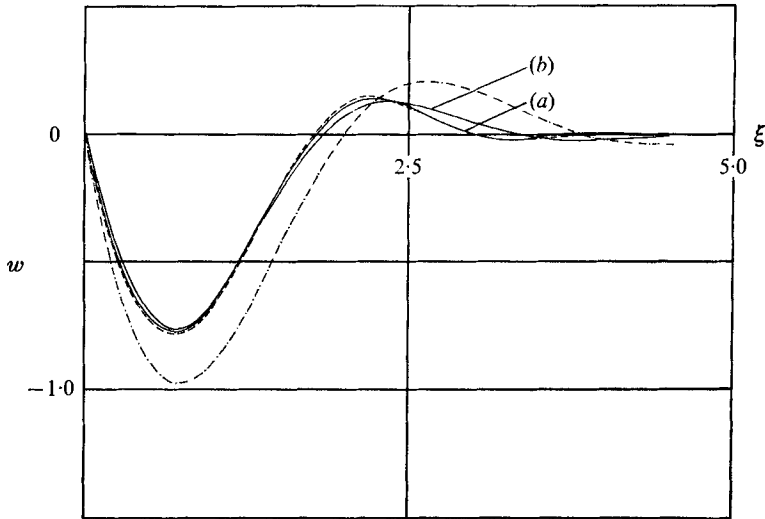


FIGURE 3. $E^{1/2}$ -layer, $\delta = +0.5$. - - -, numerical solution. (a) Two-term asymptotic solution. (b) Two-term Nakayama & Usui (1974) solution. ·-·-, incompressible case.

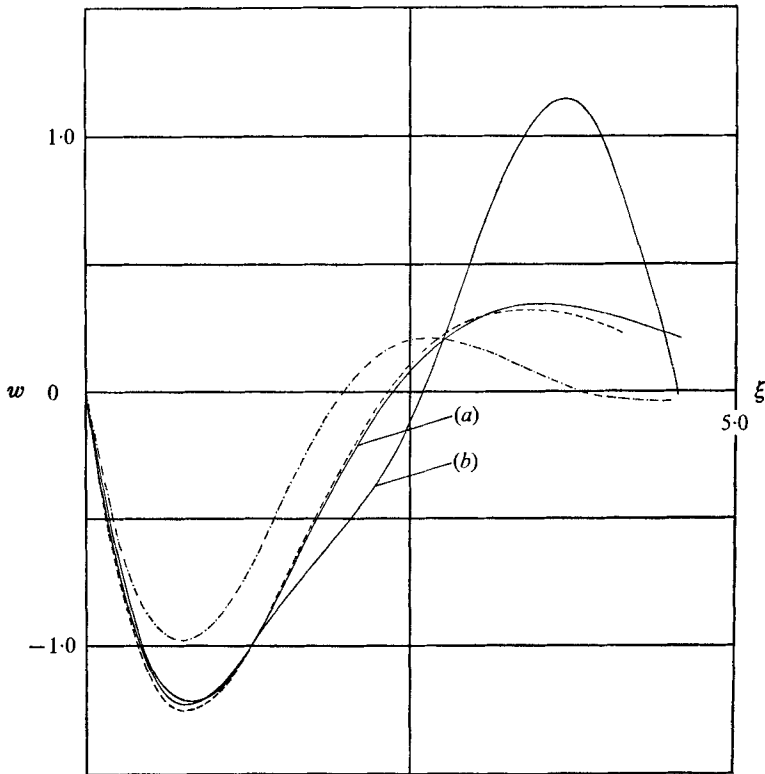


FIGURE 4. $E^{1/2}$ -layer, $\delta = -0.5$. - - -, numerical solution. (a) Two-term asymptotic solution. (b) Two-term Nakayama & Usui (1974) solution. ·-·-, incompressible case.

In figures 3 and 4 numerical (see §7) and two-term asymptotic solutions for the axial velocity in the $E^{\frac{1}{2}}$ -layer are shown for $\delta = \pm 0.5$. $\alpha^2 r_0^2 = 0.2$ in all cases. The corresponding Nakayama–Usui (1974) solutions are also shown in these graphs. For these values of δ it turns out that adding extra terms to the asymptotic solution gives only minor improvements. It can be seen from figure 3 that, perhaps somewhat surprisingly, both the Nakayama–Usui solution and the present asymptotic solution agree well with the numerical solution for $\delta = 0.5$. However for $\delta = -0.5$, as shown in figure 4, the non-uniformity for large ξ in the Nakayama–Usui solution shows up. In this case the asymptotic solution starts bending away from the numerical solution near $\xi = 4$, which indicates that a ‘super-slow’ variable $\delta^2 \xi$ is needed for an accurate asymptotic solution for values of ξ larger than 4 (cf. Nayfeh 1973, p. 228). For $\delta = -0.25$ (not shown here) the Nakayama–Usui solution is acceptable. For the curves in these figures seven terms in the rather poorly converging sum (24) were computed and nonlinear extrapolation was used for the summation (Shanks 1955). Numerical experiments indicated that the relative error is smaller than 10^{-4} .

7. Numerical solutions

For comparison with the asymptotic solutions, numerical solutions for both layers have been computed. In order to obtain starting values for a numerical integration, a straightforward procedure would be to compute the asymptotic behaviour of the solutions at large distances from the wall for small δ . However, in order to reduce the range of the independent variable to save computer time, another route was followed. The method is to let the computer select the linear combination of the exact solutions which satisfies the correct boundary conditions at the wall and is a regular perturbation from the incompressible case at some large but finite distance (an integral number of step lengths) from the wall. The perturbation solution was computed to second order in the ratio of boundary-layer thickness to the scale height of the basic density field. The actual computation is straightforward but involves some tedious algebra and is therefore not reproduced here. Such a solution is, of course, only valid locally but this is all we need to start a numerical integration. This calculation can in principle be carried out analytically for the $E^{\frac{1}{2}}$ -layer, but the algebra becomes rather extensive and numerical integration was used instead. The procedure is heuristic and would certainly not work for strongly oscillating functions having the same order of magnitude over the interval considered. However, since the results were found from numerical experiments to depend very weakly on where the outer boundary condition is applied, the method can be expected to give reasonable results. The reason why this procedure works is, of course, that the physically insignificant solutions decay very rapidly as the numerical integration proceeds inwards to the wall. In the incompressible case these solutions will decay exponentially.

Equation (25) cannot be integrated numerically using standard methods for large values of λ and positive values of δ . The reason for this is that rapidly growing parasitic errors will contaminate the two solutions having the smallest growth rates as the numerical integration proceeds towards the wall. This pheno-

menon is well known in the theory of hydrodynamic stability. In the present work the method of orthogonalization (e.g. see Wazzan, Okamura & Smith 1967) was used to overcome this difficulty.

8. Conclusions

For density scale heights large but finite compared with $E^{\frac{1}{2}}$ and/or $E^{\frac{1}{3}}$ the effect of compressibility on vertical boundary layers in a rapidly rotating gas is to thicken these layers at an outer container wall. At an inner wall the reverse is true. For the $E^{\frac{1}{2}}$ -layer an exact solution and a multiple-scale type of asymptotic solution have been computed. These solutions agree at least to third order for a large density scale height and are in very good agreement with a numerical solution. For the $E^{\frac{1}{3}}$ -layer no exact solution could be found but the asymptotic and numerical solutions show good agreement. For the very large density scale heights the Nakayama–Usui (1974) solutions were found to give reasonable results.

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